Math 54: Topology
Proof Techniques 101

In general, mathematically proving statements requires creativity and a significant reserve of patience; there is no recipe (or collection of recipes) that will cover all the different techniques and ideas. However, there are standard methods and common themes that *can*[1] help guide proofs depending on the formulation of the statement. Here are a few[2]:

Statement Form	**Standard Proof Methods**
If A, then B.	DIRECT: Assume A and deduce B. CONTRAPOSITIVE: Prove "if not B, then not A." That is, assume not B and deduce not A.
A if and only if B.	Prove "if A, then B" and "if B, then A."
Not A.	CONTRADICTION: Assume A and deduce a contradiction. See "ON PROVING 'NOT A'."
A or B.	Prove "if not A, then B." Prove "if not B, then A." CONTRADICTION: Assume not A and not B and deduce a contradiction. Consider all possible cases and show that in some cases A holds and B holds in the rest.
For all x, $A(x)$.	Take an *arbitrary* object x and deduce $A(x)$. CONTRADICTION: Assume there exists x such that $A(x)$ is false and derive a contradiction.
There exists x such that $A(x)$.	*Find* a specific x for which $A(x)$ is true. CONTRADICTION: Assume that $A(x)$ holds for all x and derive a contradiction.
The x such that $A(x)$ is unique.	Assume there is another and deduce that they must be equal.
There is a unique x such that $A(x)$.	Prove: (1) "there exists x such that $A(x)$" and (2) "the x such that $A(x)$" is unique.
For all integers $n \geq n_0$, $A(n)$.	INDUCTION: (1) Base case: prove $A(n_0)$ is true. (2) Inductive step: prove "if $A(n)$, then $A(n+1)$."

[1] This does not mean that they *should* be used.

[2] This has been adapted from Marcia Groszek's *Some Proof Principles* (Winter 2014) and Jennifer Bowen's *Some Notes on Proof Techniques* (Fall 2014).

Commentary

On the previous page, the word "deduce" appears repeatedly. For mathematicians, deduction involves **logically arriving at the conclusion starting from your assumptions** (and only using things that follow *from* those assumptions!).

Quantifiers: Special care should be taken with proofs involving the quantifiers "for all" or "there exists."

- For all ($\forall$): It is tempting to use a handful of examples (or 10,000,000) and claim the result based on that evidence. However, the result *must* be shown for an arbitrary object.
- There exists ($\exists$): It is enough to exhibit a single example with the desired property (in topology, these are often bizarre).

On proving "not A": Disproving statements (often termed "finding a counterexample") depends greatly on the statement. For instance:

- Disproving "all men are blue" involves finding a single man who is not blue.
- Disproving "if an animal wears clothes, then it is a human" requires finding an animal wearing clothes that is not a human.
- Disproving "there (currently) exists a living cat with six legs" requires examining *every* living cat and checking how many legs they have.

Induction: Proofs by induction frequently cause trouble initially. The idea is very elegant: we setup the first rung of a ladder and we use it to build the next and so on. This technique has two essential steps:

(1) "Checking the base case": In this step, we are making sure that the statement has a chance of being true. Generally this is the easiest part of the proof.

(2) "The inductive step": This is the "use the rung you just built to make the next one" step. We assume the *inductive hypothesis* $A(n)$ and we want to use it to deduce $A(n+1)$.

We will prove "$2^n \geq n+1$ for all $n \in \mathbb{Z}_+$" by induction.

Proof. We will proceed by induction. Let $A(n)$ be the statement "$2^n \geq n+1$."

Base case: Consider $n = 1$. Then $2^n = 2^1 = 2 = 1 + 1 = n + 1$. Thus $A(1)$ holds.

Inductive step: Assume $A(n)$ for some $n \in \mathbb{Z}_+$ (i.e., $\boxed{2^n \geq n+1}\,(*)$ for this specific n). We will show $A(n+1)$. Consider

$$2^{n+1} = 2^n \cdot 2 \overset{(*)}{\geq} (n+1) \cdot 2 = 2n + 2.$$

Since n is a positive number, we have $2n > n$. Hence

$$2^{n+1} \geq 2n + 2 > n + 2 = (n+1) + 1.$$

Thus $2^{n+1} > (n+1) + 1$. Therefore $2^{n+1} \geq (n+1) + 1$ or, equivalently, $A(n+1)$ holds.

Conclusion: By induction, $A(n)$ is true for all $n \in \mathbb{Z}_+$ (which is what we wanted to show). $\square$